Question 3

Gravitational lensing is a phenomenon where light from a distant source may be deflected by the curvature of space-time caused by a massive lensing object close to or in the line of sight between an observer and a distant object. This was first directly observed during the solar eclipse of 1919 where the observed positions of stars behind the sun differed from their astrometric positions following Einstein's earlier predictions.

In the case where the observer, lensing object of mass M and source are on a straight line, light from the source is deflected by an angle α (in radians) given by

$$\alpha = \frac{4GM}{r_E c^2}$$

where G is the gravitational constant $(6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}})$, c is the speed of light $(3.00 \times 10^{8}\ \mathrm{m\,s^{-1}})$, and r_E is the Einstein radius which is the least distance between the lensing object and the apparent light path.

(a) **(4 points)** Draw a diagram to describe the physical layout of an ideal (observer, lens and point source in a straight line) lensing system. Draw the light path and mark the quantities α and r_E. Also mark the angular Einstein radius θ_E (the angular deflection of the source image as seen from earth), and the other quantities that an observer on earth can measure.

(b) **(2 points)** Sketch the image of the source (such as a star), as seen by an observer on earth, in the case where the source, lensing object and observer are on a straight line.

(c) **(3 points)** Sketch the image of the source (such as a star), as seen by an observer on earth, in the non-ideal case where the source, lensing object and observer are not in a straight line. Sketch the source-lens system to explain why this is so.

Gravitational lensing has been proposed as a method to detect massive compact halo objects (MACHOs) in our galaxy, which may be a candidate for dark matter. These objects are often dark stellar remnants such as neutron stars and black holes. As stars and MACHOs orbit in the galaxy, there is a chance that a lensing event may occur when a black hole or neutron star passes in front of a background star.

(d) **(3 points)** The Schwarzschild radius of a black hole defines the point of no return. A correct expression for the Schwarzschild radius can be obtained by taking it to be the radius where the escape speed is equal to the speed of light. This means that something inside the Schwarzschild radius cannot escape the black hole.
Using Newtonian mechanics, derive the formula for the escape speed at a distance r away from a point object of mass M. Hence, derive the Schwarzschild radius for a point object of mass M in terms of the gravitational constant G and the speed of light c. Show your steps and reasoning clearly. (This happens to give the correct expression for the Schwarzschild radius that comes from general relativity.)

(e) **(1 point)** In the case where the source, lens and observer are in a straight line, given a measurement of α and r_E, how would you calculate the Schwarzschild radius of the lensing object?

(f) **(2 points)** Consider the case where we have a lensing object of the order of a few solar masses ($M \sim$ a few $\times\ 10^{30}$ kg) in the nearby regions of the galaxy (distance $D_L \sim$ a few $\times\ 10^{18}$ m away) and a source object somewhat further out ($D_S \sim$ a few $\times\ D_L$). Which of the following apply in this case?
Choose the following conditions that apply to the case as described in the question:

- α is large and $\tan\alpha$, $\sin\alpha$, $\cos\alpha$ must be calculated exactly
- α is small and the small angle approximations to $\tan\alpha$, $\sin\alpha$, $\cos\alpha$ are permissible
- α is irrelevant and need not be calculated
- θ_E is large and $\tan\theta_E$, $\sin\theta_E$, $\cos\theta_E$ must be calculated exactly
- θ_E is small and the small angle approximations to $\tan\theta_E$, $\sin\theta_E$, $\cos\theta_E$ are permissible
- θ_E is irrelevant and need not be calculated

(g) **(3 points)** Using the conditions in part (f), rewrite your expression in part (e) in terms of measurable quantities (which are θ_E, D_S and D_L) for a lensing object of the order of a few solar masses ($M \sim$ a few $\times\ 10^{30}$ kg) and in the nearby regions of the galaxy (distance $D_L \sim$ a few $\times\ 10^{18}$ m away) with a source object somewhat further out ($D_S \sim$ a few $\times\ D_L$). Show your working.

(h) **(2 points)** Suppose we have an event where a lensing object of 6.0×10^{30} kg (3.0 solar masses), 2.6×10^{18} m away from earth passes in front of a star 9.2×10^{18} m away from earth. This happens such that the ideal configuration occurs during the event. What is the angular Einstein radius θ_E (as seen from earth) during this event when the source, lens and observer line up?